Physics 566: Quantum Optics
Problem Set #5
Due Friday Oct. 29, 2010

Problem 1: Boson Algebra (10 Points)
This problem is to give you some practice manipulating the boson algebra. A great source is the classic "Quantum Statistical Properties of Radiation", by W. H. Louisell, reprinted by "Wiley Classics Library", ISBN 0-471-52365-8.

(a) Gaussian integrals in phase-space are used all the time. Show that

$$\int \frac{d^2\beta}{\pi} e^{-A|\beta|^2} e^{\alpha\beta^* - \beta\alpha^*} = \frac{1}{A} e^{-|\alpha|^2/A}.$$

(b) Prove the completeness integral for coherent states

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \hat{1} \text{ (Hint: Expand in number states).}$$

(c) The "quadrature" operators in optics are the analogs of Q and P, $\hat{a} = \hat{X}_1 + i\hat{X}_2$. Show

$$\begin{aligned} \hat{U}^\dagger(\theta)\hat{X}_1\hat{U}(\theta) &= \cos\theta\,\hat{X}_1 + \sin\theta\,\hat{X}_2 \\ \hat{U}^\dagger(\theta)\hat{X}_2\hat{U}(\theta) &= \cos\theta\,\hat{X}_2 - \sin\theta\,\hat{X}_1 \end{aligned}, \quad \text{where} \quad \hat{U}(\theta) = e^{-i\theta\hat{a}^\dagger\hat{a}}.$$

Interpret in phase space.

(d) Prove the group property of the displacement operator

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta)\exp\{i\,\mathrm{Im}(\alpha\beta^*)\}$$

(e) Show that the displacement operators has the following matrix elements

Vacuum: $\langle 0|\hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2}$

Coherent states: $\langle \alpha_1|\hat{D}(\alpha)|\alpha_2\rangle = e^{-|\alpha+\alpha_2-\alpha_1|^2/2} e^{i\,\mathrm{Im}\left(\alpha\alpha_2^* - \alpha_1\alpha^* - \alpha_1\alpha_2^*\right)}$

Fock states: $\langle n|\hat{D}(\alpha)|n\rangle = e^{-|\alpha|^2/2}\mathrm{L}_n\left(|\alpha|^2\right)$, where L_n is the Laguerre polynomial of order n

Problem 2: The Wigner Function

The Wigner function for a single mode of the field described by a state $\hat{\rho}$ can be understood as the expectation value of a Hermitian operator,

$$\hat{W}(\alpha) = \frac{1}{\pi^2} \int d^2\beta \, \hat{D}(\beta) \exp\left(\alpha\beta^* - \alpha^*\beta\right),$$

$$\text{so that, } W(\alpha) = \left\langle \hat{W}(\alpha) \right\rangle = \frac{1}{\pi^2} \int d^2\beta \, Tr\left(\hat{\rho}\hat{D}(\beta)\right) \exp\left(\alpha\beta^* - \alpha^*\beta\right).$$

(a) Show that $\hat{W}(\alpha) = \hat{D}(\alpha)\hat{W}(0)\hat{D}^\dagger(\alpha)$.

Consider, $\hat{W}(0) = \frac{1}{\pi^2} \int d^2\beta \, \hat{D}(\beta) = \frac{1}{\pi^2} \int dX_0 dP_0 \, \hat{D}(X_0, P_0)$, with $\hat{D}(X_0, P_0) = \exp\left[-2i\left(X_0\hat{P} - P_0\hat{X}\right)\right]$.
To determine $\hat{W}(0)$, consider the position representation.

(b) Show that the matrix element of the displacement operator in the position representation is $\langle X' | \hat{D}(X_0, P_0) | X \rangle = \exp[iP_0(X_0 + 2X)] \delta(X' - X - X_0)$.

(c) Use this to show that $\hat{W}(0) = \frac{2}{\pi} \int dX \, |-X\rangle\langle X|$, and then argue that $\hat{W}(0) = \frac{2}{\pi}\hat{\Pi}$, where $\hat{\Pi}$ is the parity operator, familiar in wave mechanics.

(d) Show that in the number-state basis, $\hat{W}(0) = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n |n\rangle\langle n|$, and thus the Wigner function at the origin is $W(0) = \frac{2}{\pi}\left\langle \hat{\Pi} \right\rangle = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n p_n$, where p_n is the probability of finding n photons in the mode.

The above result shows that one can measure the Wigner function of a mode by counting the number of photons in many copies, collecting statistics, and determining the p_n.

(e) To get the Wigner function at any other point α, one need first displace the state away from the origin as in part (a), and measure the mean value of parity through photon counting on the displaced state.
Show: $W(\alpha) = \frac{2}{\pi}\left\langle \hat{D}(\alpha)\hat{\Pi}\hat{D}^\dagger(\alpha) \right\rangle = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n p_{n\alpha}$, where $p_{n\alpha} = \langle n, \alpha | \hat{\rho} | n, \alpha \rangle$, $|n, \alpha\rangle \equiv \hat{D}(\alpha)|n\rangle$.

Problem 3: A "Schrödinger cat" state.

Consider a superposition state of two "macroscopically" distinguishable coherent states, $|\psi\rangle = N(|\alpha_1\rangle + |\alpha_2\rangle)$, $|\alpha_1 - \alpha_2| >> 1$, where $N = \left[2\left(1+\exp\{-|\alpha_1-\alpha_2|^2\}\right)\right]^{-1/2}$ is normalization. This state is often referred to as a "Schrodinger cat", and is very nonclassical.

(a) Calculate the Wigner function, for the simpler case $|\psi\rangle = N(|\alpha\rangle + |-\alpha\rangle)$, with α real, and plot it for different values of $|\alpha_1 - \alpha_2| = 2\alpha$. Comment please.

(b) Calculate the marginals in X_1 and X_2 and show they are what you expect.

Problem 4: Thermal Light

Consider a single mode field in thermal equilibrium at temperature *T*, Boltzmann factor $\beta = 1/k_B T$. The state of the field is described by the "canonical ensemble",

$\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}}$, $\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a}$ is the Hamiltonian and $Z = Tr\left(e^{-\beta \hat{H}}\right)$ is the partition function.

(a) Remind yourself of the basic properties by deriving the following:

- $\langle n \rangle = \frac{1}{e^{\beta\hbar\omega} - 1}$ (the Planck spectrum)
- $P_n = \frac{\langle n \rangle^n}{(1+\langle n \rangle)^{n+1}}$ (the Bose-Einstein distribution). Plot a histogram for various $\langle n \rangle$.
- $\Delta n^2 = \langle n \rangle + \langle n \rangle^2$. How does this compare to a coherent state?
- $\langle \hat{a} \rangle = 0 \Rightarrow \langle \vec{E} \rangle = 0$. How does this compare to a coherent state?

(b) Find the *P. Q,* and *W* distributions for this field, and show they are *Gaussian* functions. For example, you should find $P(\alpha) = \frac{1}{\pi\langle n \rangle}\exp\left(-\frac{|\alpha|^2}{\langle n \rangle}\right)$. Show that these three distributions give the proper functions in the limit, $\langle n \rangle \to 0$, i.e. the vacuum.

(c) Calculate $\Delta n^2, (\Delta X_1(\theta))^2, (\Delta X_2(\theta))^2$ using an appropriate quasi-probability distribution. Interpret Δn^2 as having a "particle" and a "wave" component.